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# OPTIMIZATION MODELS FOR PLANNING ECONOMIC DEVELOPMENT

by

SILVIA PARIENTE

Technical Report No. 130

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# FOR PLANNING ECONOMIC DEVELOPMENT

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SILVIA PARIENTE

Technical Report No. 130

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Contract N00014-75-C-0556, Office of Naval Research
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NR 347-027 M.I.T. OSP 82491

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## FOREWORD

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Jeremy F. Shapiro
Acting Director

## ABSTRACT

In recent years, more and more countries have experimented with quantitative methods as a way to design short term and long term plans, and to evaluate the impacts of investment and other policies on the future development of their economies. Optimization models have proved useful in several fields of economics, such as economic growth and development planning, urban and regional economics, agricultural and energy economics, etc. The purpose of this paper is to formulate optimization models that can be applied fruitfully for economy-wide planning, sectoral planning, and project evaluation. It is an attempt at synthesizing the different models encountered in the literature, and at describing some of the difficulties inherent to this approach.

## 1. INTRODUCTION

Governments have the responsibility of formulating economy-wide policies; they also make decisions at the level of the sector and the project. This is true even in countries where the private sector is very important. Governments have to design policy to motivate private and public decision makers to act in a way consistent with the stated goals. The role of planning is to determine realistic goals and a way to achieve them. Basic factors for economic development are, for instance, capital formation and financing, skill formation, use of external resources for capital and imports, reallocation of resources to avoid bottlenecks.

Planning used to be done intuitively, but, more and more, analytical techniques are used: econometric models and planning models. Mathematical programming techniques provide the necessary tools to solve these planning models. These can be employed to analyze the relationsips between certain strategic elements of a plan and to study the consequences of changing the underlying assumptions.

Models are specialized according to economic function, sector, region or time period. Hence the problem of economic development can be approached from several different viewpoints. The first approach is macroeconomic growth and the selection of an economic growth rate consistent with domestic policy and foreign aid. Models used for this purpose range from aggregate ones to detailed multi-sector models. Economy-wide models

are reviewed in section 2. When more detailed information on a particular sector is needed, an economy-wide model is impractical - it would be highly disaggregated and costly to use-and in such cases a sectoral model is preferred. If a particular set of decisions involves policies and projects mainly internal to a given sector, little is lost by using a model of the sector rather than one of the whole economy. For all these reasons, sectoral models are used extensively in agriculture, manufacturing sectors, energy and transportation sectors, and, even, education. The general characteristics of these models are described in Section 3. Economic planning also involves consideration of alternative public investments under project analysis. Traditionally, this is done using cost-benefit analysis on a single project basis. This approach is not satisfactory when the impact of the project on the rest of the economy is very important and interdependencies have to be modelled. Other approaches should, in this case, be preferred. These are discussed in Section 4. Regional planning is concerned with the same type of decisions - at macro, sectoral and project level - when the scope of the policies is the region rather than the entire economy. The role of multi-level planning is to make all these models at different levels consistent. Sectoral planning should be consistent with macro planning; the same is true of regional planning. A hierarchical approach to economic planning is used. This is achieved by using decomposition techniques, either formally or informally.*

* For more on multi level planning, see [16], [17], Duloy in [6], [23], [25].

## 2. ECONOMY-WIDE MODELS

As was said before, economy-wide models range from simple aggregate models to detailed multi-sector ones. This survey does not deal with aggregate models of optimal growth theory,* nor with multi-sector consistency models, but only with multi-sector optimization models.** These should be detailed enough to be operational. They should be solved numerically to provide forecasts based on real data, hence they should keep to a manageable size (or else special solution techniques, such as decomposition, have to be used). In fact, the consistency approach is contained in the optimizing approach. The algebraic equations of a consistency model become the constraints of an optimization model. Some of the variables, exogenous in the consistency model, become endogenous in the optimization model. A welfare function to be maximized is specified to allow the user to choose among different consistent plans. Hence an optimization approach seems to be preferable as it contains the consistency approach in itself, and the dual variables, or shadow prices, of an optimization model contain useful information. Some models incorporate explicitly the time factor, others are static.

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* See Domar [9] and Rosenstein-Rodan [31].

** For a complete review of economy-wide models, see Taylor in [2] and, for multi-sector models see Manne [26].

## 2.1 Static Linear Programming Models

Multi-sector models are usually built around an inter-industry flow table. The basic supply-demand balance equation in input-output analysis is:

$$x_i + m_i = \sum_{j=1}^{n} x_{ij} + c_i + g_i + j_i + e_i + s_i$$

where

$i = 1....n$

$x_i$ = volume of gross output for sector i

$m_i$ = competitive imports into sector i

$x_{ij}$ = intermediate sale from sector i to sector j

$c_i$ = consumer demand for product of sector i

$g_i$ = government expeditures for sector i products (investment by sector of origin)

$j_i$ = capital formation demand for sector i products (investment by sector of origin)

$e_i$ = exports from sector i

$s_i$ = change in stocks for sector i products

$c_i + g_i + j_i + e_i + s_i = f_i$ = total final demand for sector i

An implicit assumption is that each sector produces a single good available interchangeably for consumption, investment, export, etc. Another basic assumption is that $x_{ij} = a_{ij}x_j$, where $a_{ij}$ is a fixed technological coefficient. Then, if we let $A = (a_{ij})$ be the consumption matrix,

$$x + m = Ax + f$$

$$(I-A)x + m = f$$

(I-A) is the net production matrix. The first set of constraints in a static LP planning model are these input-output relations rewritten as inequalities (uses $\leq$ availabilities) and the most usual maximand is aggregate consumption c. The first constraints are then:

(1) $x \geq Ax + \xi c + j$

where the only final demands here are the vectors of investment levels by origin j and of private consumption $\xi c$. Here the distribution among different consumption goods is fixed and is exogenous to the model. In other models,* the choice between consumption goods is endogenous. The investment sector can be treated in several alternative ways: j can be fixed exogenously, or a dynamic element can be introduced in the model (investment during the plan period provides capacity for the post plan period) and j is linked to x by $j = \gamma Bx$, where B is the capital coefficient matrix (we return to this definition in the next paragraph) and $\gamma$ is the so-called stock flow conversion factor. It can also be interpreted as the rate of growth of output in the post plan period.**

The next set of constraints is for bounds on factor use. They include, for instance, labor requirements, capacity constraints, etc. They can be written: $Fx \leq \bar{s}$; where $F = (f_{ij})$

---

* See [15], [17].

** e.g., see Manne [26], Bruno [4], Taylor in [2], Ginsburgh and Walbroeck [15].

is an n x n matrix; $f_{ij}$ is the amount of factor i needed to produce one unit of commodity j. The basic economy-wide linear programming model is then:

$$
\begin{aligned}
&\text{(MSP)} \quad \text{Max } c \\
&\quad \text{s.t. } (I-A)x \geq \xi c + Bx && (1) \\
&\quad\quad Fx \leq \bar{s} && (2) \\
&\quad\quad c,\ x \geq 0
\end{aligned}
$$

It can be extended by adding other elements in the vector of final demand (right hand side of (1)) - variables such as exports or imports, or exogenous elements such as government expenditures - and other constraints such as for foreign exchange, if imports and exports are included, or political constraints.

Let q and p respectively be the vectors of shadow prices for (1) and (2). The optimality conditions for (MSP) are:

$$
\begin{aligned}
&(I - A - \gamma B)x - \xi c \geq 0; && [(I - A - \gamma B)x - \xi c]q = 0 && (3) \\
&Fx - \bar{s} \leq 0 && (Fx - \bar{s})p = 0 && (4) \\
&-q(I - A - \gamma B) + pF \geq 0; && [-q(I - A - \gamma B) + pF]x = 0 && (5) \\
&q\xi - 1 \geq 0; && (q\xi - 1)c = 0 && (6) \\
&x,\ c,\ p,\ q \geq 0
\end{aligned}
$$

They can be interpreted in the following way, starting with (6). In an optimal solution, c is positive (assuming $\bar{s} > 0$), then (6) is $q\xi = 1$, which says that the marginal cost of the consumption activity is equal to 1, its marginal utility in the objective function ($q_i$ is the marginal cost of producing

good i and $\xi_i$ is the marginal consumption share of good i). (5) can be written

$$q \leq qA + q\gamma B + pF$$

or price $\leq$ cost (production cost + rents)

It is the non-profit condition, and the output of a good is positive only if profit is equal to cost. If $(I - A - \gamma B)$ is a leontief matrix, then all the elements of $(I - A - \gamma B)^{-1}$ are positive and the positivity of c implies the positivity of x (assuming $\xi > 0$). In that case (5) becomes an equality. The interpretation of (4) is that the cost of a primary factor is positive only if that primary factor is not in surplus. (3) says that the cost of producing good i is positive only if net output of good i is equal to consumption of good i.

The optimality conditions for extended models have similar interpretations. They are not reproduced here.*

The size of these models is not usually very large and they can be solved efficiently by any linear programming technique. Several experiments can be conducted at a reasonable cost, using sensitivity analysis and parametric programming. Shadow prices are provided along with the primal solution and they contain useful information, as can be seen from the interpretation of the optimality conditions. They can be useful

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* See Taylor in [2].

for project appraisal. We shall return to this question in Section 4.

The main drawback of these models is the linearity of the objective function which can create erratic behavior of the solution (e.g., too much investment in one sector and not enough in others). This problem will be discussed in more detail in the following subsection for dynamic models. Some of the remarks there are applicable here.

## 2.2 Dynamic Linear Programming Models

They have the same basic characteristics as static LPs, but capital formation is regarded now as a means of increasing future productive capacities and is endogenous.* Successive periods are linked through capital accumulation equations. The other constraints are simply repeated for each time period. All the variables are subscripted by a time index. The vector of investment demands by origin j is related to the vector of investment demands by destination i by the following relationship: $j = Bi$ (assuming infinite lifetime for capital), where B is a distribution matrix for investment demands (or capital coefficient matrix):

$B = (b_{ij})$ n x n matrix

$b_{ij}$ = quantity of investment goods to be produced by sector i per unit of net capacity increase in sector j.

---

* Skill formation could also be an endogenous activity, e.g., see Blitzer [3], Goreux [17] and Manne [27].

If the length of the time periods is assumed to be the same as the gestation lags in investment, then:

$$i_t = k_{t+1} - k_t$$

A more complex structure for gestation lags can be introduced through more complex relations (see Eckaus and Parikh [11]). The material balance equations become:

$$x_t \geq Ax_t + \xi_t c_t + Bi_t \qquad (1)$$

The full model is:

$$\text{(MDP)} \qquad \text{Max} \sum_t \omega^{t-1} c_t$$

$$(I - A)x_t \geq \xi_t c_t + Bi_t \qquad \text{For all t.} \quad (1)$$

$$i_t = k_{t+1} - k_t \qquad \text{For all t.} \quad (2)$$

$$Kx_t \leq k_t \qquad \text{For all t.} \quad (3)$$

$$Fx_t \leq s_t \qquad \text{For all t.} \quad (4)$$

$$x_t,\ c_t,\ i_t \geq 0$$

$$k_1 = \bar{k}_1 \text{ given}$$

Constraints (3) are the capacity constraints; K is an n x n diagonal matrix of sectoral capital-output ratios. The other constraints have been interpreted before. Constraints (1), (2) and (3) could alternatively be written:

$$(I - A)x_t \geq \xi_t c_t + B(k_{t+1} - k_t) \qquad \text{For all t.} \quad (1)$$

$$k_{t+1} \geq k_t \qquad \text{For all t.} \quad (2)$$

$$Kx_t \leq k_t \qquad \text{For all t.} \quad (3)$$

or

$$(I - A)x_t \geq \xi_t c_t + Bi_t \qquad \text{For all t.} \quad (1)$$

$$Kx_t \leq \bar{k}_1 + \sum_{\tau=1}^{t-1} i_\tau \qquad \text{For all t.} \quad (3)$$

This last formulation will be adopted. The objective function is the discounted sum of aggregate consumption; $\omega$ is a discount factor $0 < \omega \leq 1$. In the objective function, the range of the index t in the sum purposely has not been specified. Ideally, it should run from t=1 to infinity, this system of equations having no natural end. But an infinite horizon detailed model is impractical. Only very simplified infinite horizon models can be solved analytically. If we want to solve the model numerically, a finite planning horizon T has to be chosen. Then terminal conditions have to be specified to take into account the post plan period. This involves adding constraint to (MDP) and/or another term in the objective function. The planning horizon T and the terminal conditions sould be such that they do not affect too much the decisions in the first years of the plan. The problem of terminal conditions is discussed in more detail in the next subsection.

We now interpret the optimality conditions for (MDP). They decompose neatly into within- and between-periods classes. Let $q_c$, $r_t$ and $p_t$ be the vectors of dual variables for (1), (3) and (4) respectively in each time period t.

$$(I - A)x_t - \xi_t c_t - Bi_t \geq 0 \qquad [(I - A)x_t - \xi_t c_t - Bi_t]q_t = 0 \quad (5)$$

$$Kx_t - \sum_{\tau=1}^{t-1} i_\tau - \bar{k}_1 \leq 0 \qquad (Kx_t - \Sigma i_\tau - \bar{k}_1)r_t = 0 \quad (6)$$

$$Fx_t - \bar{s}_t \leq 0 \qquad (Fx_t - \bar{s}_t)p_t = 0 \quad (7)$$

$$-q_t(I - A) + r_t K + p_t F \geq 0 \qquad (-qt(I - A) + r_t K + p_t F)x_t = 0 \quad (8)$$

$$q_t \xi_t - 1 \geq 0 \qquad (q_t \xi_t - 1)c_t = 0 \quad (9)$$

$$q_t B - \sum_{\tau=t+1} r_t \geq 0 \qquad (q_t B - \Sigma r_\tau)i_t = 0 \quad (10)$$

The interpretation of (5), (7), (8) and (9) for each time period is the same as for the corresponding conditions in the static case. (6) and (10) are the between-period conditions. (10) says that the cost of capital for producing good i in period t must be greater than or equal to the sum of the rents for period t+1 on, with equality holding if investment is positive in period t. If $c_t$ is positive in each period t then $x_t$ is positive in each period and (8) and (9) hold with equality. But this will not be true in general. These linear programming models usually display the so-called "flip-flop" behavior, i.e, consumption is concentrated in the last periods of the plan and investment in the first periods, the period in which the change occurs depends on the discount factor $\omega$ (this behavior is illustrated in Appendix I). This behavior is due to the linearity of the objective function. Other characteristics of the solution of such models is extreme specialization in investment expenditures in some sectors. This is why dynamic models with linear maximands contain some kinds of smoothing devices, such as constraints on consumption - lower bound on consumption in each time period or specification of a minimum growth rate for consumption (e.g., see Ekaus and Parikh [11]) - on savings, on investment - upper bound on total investment in each period; this restriction, imposed for technical reasons, reflects in fact decreasing returns to investment. These artificial devices obscure the dual solution of the model and limit intertemporal trade-offs. In fact, the solution to all these difficulties lies in the use of a non-linear welfare

function with decreasing marginal utility, i.e., a strictly concave objective function. Such objective functions are presented along with alternative objective functions in paragraph 2-4.

2.3 Terminal Conditions

Whenever dealing with a dynamic model, there arises the problem of how to truncate the planning horizon and to set terminal conditions so as not to affect too much the results obtained for the plan period. This is a difficult problem since investments during the terminal year do not affect available capacities within the planning horizon. It will be optimal to invest in the terminal year only if capacity created by these investments is valued one way or another. As rollover planning is generally used, terminal conditions do not have to be completely accurate but they should not affect the first years of the plan. This is difficult as an erratic solution in the terminal year can create erratic behavior in the initial years, since the choice between consumption and investment is made simultaneously for all years of the plan.

There is no general agreement on how to set terminal conditions but several possibilities exist. One truncation procedure may be described as follows: it is assumed that consumption will grow in the post plan period (after year T) at some predetermined growth rate $\gamma$; this determines the growth rate

of output g in the post plan period;* these growth rates $\gamma$ and g could be dictated by the solution of an aggregate growth model. Then the objective function consists of two components, one for the plan period and one for the post plan period:

$$\sum_{t=1}^{T}\omega^{t-1}c_t + \sum_{t=T+1}^{\infty}\omega^{t-1}(1+\gamma)^{t-T}c_T = \sum_{t=1}^{T}\omega^{t-1}c_t + \frac{\omega^T(1+\gamma)}{1-\omega(1+\gamma)}c_T$$

Hence, post plan consumption is valued in the objective function. Investment in the terminal year must be sufficient to create enough capacity to sustain the growth; a constraint on terminal investment is included. This procedure leads to satisfactory results if the specified growth rate is close enough to the asymptotic growth rate. Such an asymptotic growth rate cannot be computed algebraically for complex models and it has to be approximated. This procedure is used by Goreux [17] and Eckaus and Parikh [11].

Another trucation procedure, simpler and popular among model builders, is the following: the level of terminal capital stock, or terminal investment, is specified *a priori* and the model is solved with this constraint added; equivalently terminal capital stock can be valued in the objective function at predetermined prices. The horizon effects are not too important if these values are close to those that could be described with a non-truncated model. But the quantity of terminal capital stock as well as its price is hard to evaluate and this

* This procedure, in fact, is equivalent to specifying that the economy should reach an optimal balanced growth path at the end of the planning horizon.

can create difficulties. For instance, if the price selected for terminal capital stock is too high, with a linear maximand, it will be optimal to invest everything in the terminal year and, if it is too low, everything is consumed. This can affect the solution during the initial years as well. Probably the best is to use some kind of iterative procedure where these values are guessed and revised until a satisfactory solution is obtained.

It has also been suggested to tie investment level to consumption level in the terminal year of the plan, to avoid the all-investment or all-consumption behavior.

In any case, to lesser horizon effects, it is best to solve the model with a planning horizon T larger than the one the planner is interested in.

## 2.4 Objective Function

If an optimization model is preferred to a consistency model, a welfare function to be maximized has to be chosen. In the previous paragraphs, this function was linear but we saw that it leads to an erratic solution ("bang-bang"). The alternative is to specify a welfare function with decreasing marginal utility. A very popular one is the isoelastic cardinal utility function. This function is assumed to be additively separable over time and the maximand is the discounted utility of consumption:

$$\sum_{t=1}^{\infty} \omega^{t-1} \frac{c_t^{1-\upsilon}}{1-\upsilon}$$

where $c_t$ is aggregate consumption and $\upsilon$ is the utility elasticity parameter ($\upsilon \geq 0$, $\upsilon \neq 1$). If $\upsilon=0$, the utility function is linear. For $\upsilon=1$, we obtain a logarithmic utility function. For $\upsilon>0$, this function is strictly concave. If the choice among consumption goods is endogenous to the model, the utility function must be further disaggregated and it is a function of consumption of each good rather than a function of aggregate consumption. It is usually assumed to be additively separable over commodities and the maximand is:

$$\sum_{t=1}^{\infty} \omega^{t-1} [\sum_i U_{it}(c_{it})] \qquad (2)$$

where $c_{it}$ is consumption of good i in time period t. The utility function $U_{it}$ can also be of the isoelastic type:

$$U_i(c_i) = A_i \frac{c_i^{1-\upsilon_i}}{1-\upsilon_i} \qquad (2)'$$

where

$\upsilon_i = \frac{1}{\eta_i \sigma}$ $\eta_i$ = income elasticity

$\sigma$ = overall elasticity of substitution

These parameters can be estimated from quantity and price data.

The utility functions (1) and (2) have been extensively used in theoretical work but not yet in practical models for development planning (see Manne [26]). An important exception is the work of Goreux [17] who estimated and used as maximand a utility function such as (2) and (2)'.

Some authors adopt the "gradualist consumption path" approach (e.g., see Manne [27]). Aggregate consumption is assumed to follow a "gradualist" path over time and increments in consumption grow at a predetermined rate $\gamma$: $c_{t+1} - c_t = (1 + \gamma)(c_t - c_{t-1})$; $\gamma$ is the asymptotic rate of growth of consumption. The maximand can be any of the variables $c_t$ $1 \leq t \leq T$.

The objective functions discussed before allow for some intertemporal trade-offs by taking into account the values of consumption in all the years of the plan period. Other objective functions depend only on the terminal value of some variable. For instance, it has been suggested to use as maximand terminal GDP or GNP (see Blitzer [3]), terminal consumption or terminal capital stock.

If the goal is self reliance, then trade deficit at the end of the plan can be used as a minimand.

Alternative goals can dictate alternative objective functions. A few of them are listed here: if the government is concerned with unemployment, total employment can be used as maximand; if the government tries to achieve a better income distribution the population is partitioned into different income groups and the objective function is a weighted sum of isoelastic utility functions (static case):

$$\sum_{i=1}^{n} W_i \frac{c_i^{1-\upsilon}}{1-\upsilon}$$

or

$$\sum_{i=1}^{n} W_i \log c_i$$

where there are n income classes (n=3, for instance, rich, middle and poor), $c_i$ is aggregate consumption of the $i^{th}$ group; the weights $W_i$ can be adjusted to reflect different preferences (for more on this see Chenery [6]).

In conclusion, there are a lot of alternative possible objective functions for multi-sector models for development planning. Depending on the situation, one may be more appropriate than another but the choice is not always easy. Several objectives may be relevant to the problem at hand and these different goals are usually conflicting. In that case a multi-criterion optimization could be used.*

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* For a review see Loucks in [2].

## 3. SECTORAL MODELS

### 3.1 Introduction

While economy-wide models attempt to cover all the sectors of the economy from a macroeconomic viewpoint, sectoral models cover only one sector of the national economy (for instance the energy sector) or even sometimes only part of a sector (such as the electric power industry, which is a subset of the energy sector). The use of a sectoral model is dictated by the need for a more detailed analysis. Hence, as was noted by Duloy in [6], they can be disaggregated in one or more of the following dimensions: spatially, allowing for inter-regional analysis; temporally, with the length of the time periods usually smaller in a sectoral model than in an economy wide model; by product definition; by technology for producing a given product, allowing for substitutability; by source and quality of primary factors (for instance, different skill levels for labor); and by size of producing units. Depending on the formulation adopted, they can be useful in deciding, for example, the timing, location and scale of investments within individual sectors or in evaluating the impacts of some policy instruments, such as taxes, subsidies, quotas, etc. Sectoral models are in fact very diverse. They range from aggregate macro-models to detailed micro-models. Depending on the situation of the sector in the economy, they can be more or less detailed. For example, a model of a complex sector in a highly industrialized economy is certainly very different from a model of a new sector in a less developed country. The first one is the aggregation of numerous different firms, or industries

and cannot contain detailed information about each one of them. In the second case, the sector may be constituted of a unique firm, even a unique plant, and the model may be more like a firm-wide analysis in a developed country, thus allowing for testing detailed investment decisions.

Even though the nature of sectoral models can be so different, it is still possible to describe some general properties.

Sectoral models, whether static or dynamic, can be classified in two broad categories: minimum cost linear programming models and non-linear equilibrium models. In the first case, the objective is to satisfy a given demand at minimum cost, the quantity available of primary resources being limited. In the second case, demand and supply are no longer fixed but rather, demand and supply functions are known, relating price and quantity demanded or quantity supplied. In that case, demands, supplies and prices are all variables in the model.

The objective is now to maximize some measure of welfare. The welfare function is usually non-linear and is a function of the demand and supply vectors. If the demand and supply functions are integrable, an acceptable welfare function is the sum of producers' and consumers' surpluses.* Let d be the vector of demands, s the fector of supplies, and p the

---

* The details on the definition of this welfare function and on the integrability conditions are given in Appendix II.

vector of prices,

$$p = \phi(d) \qquad\qquad p = \psi(s)$$

$$W(d,s) = \int_0^d \phi(\delta)d\delta - \int_0^s \psi(\xi)d\xi = f(d) - g(s)$$

For instance, if the function $\phi$ and $\psi$ are linear, the welfare function is quadratic in d and s. We would also like the welfare function to be concave; this will happen if the functions $\phi$ and $\psi$ are respectively nonincreasing and nondecreasing.*

Models of the second type correspond to a competitive equilibrium. Depending on the situation of the sector in the economy, one formulation might be more appropriate than the other. For instance, it has usually been found very useful and important to incorporate demand functions in agriculture models and thus simulate a competitive market equilibrium (e.g., see CHAC, model of the Mexican agriculture sector [10]). In other cases, it has been decided that the elasticity of demand was small and could be neglected and that the sector could be treated as a price-taker, output-taker.** Only a careful preliminary study can help in deciding which formulation to adopt

---

* i.e., for the multi product case with interrelated demand and supply functions, the Jacobian matrices for $\phi$ and $\psi$ are respectively semi-definite negative and semi-definite positive.

** This happens usually when the sector constitutes a small fraction of the entire economy; for an example, see ENERGETICOS, model of the Mexican energy sector [14], [29].

and, eventually, the approach should be validated *a posteriori* by different experiments [29]. The formulation of these models are discussed in more detail in the following paragraphs.

## 3.2 Static Models

Notations:

x: n - vector of activities

c: marginal cost of operating activities (constant)

d: m - vector of demands

A: m x n - output matrix. Ax = output

s: k - vector of supply

B: k x n - input matrix. Bx = input

A is usually a leontief matrix, i.e., each activity produces only one good but some goods may be producible by several activities.

### 3.2.1 Minimum-Cost Linear Programs

This type of formulation is obtained when the sector is assumed to be price-taker and output-taker.

$$\text{(SSLP)} \quad \begin{aligned} &\text{Min } cx \\ &Ax \geq d \\ &Bx \leq s \\ &x \geq 0 \end{aligned}$$

In this formulation, primary resources are available in limited quantity whatever the price is. But some of them could be available in infinite quantity at a given marginal cost. In that case, we obtain the following formulation:

$$\begin{aligned} &\text{Min } cx + ps \\ &Ax \geq d \\ &B^1x - s \leq 0 \\ &B^2x \leq \bar{s} \\ &x,\ s \geq 0 \end{aligned}$$

where p is the vector of marginal cost of primary commodities and the input matrix B has been partitioned

$$B = \binom{B^1}{B^2}$$

The first case corresponds to zero elasticity supply; in the second formulation, the elasticity of supply is infinite for commodities of the first group and zero for commodities of the second group.

The optimality conditions for (SSLP) are, letting q and p be the dual variables for the demand and supply constraints respectively:

$$Ax \geq d \qquad (Ax - d)q = 0 \qquad (1)$$

$$Bx \leq s \qquad (Bx - s)p = 0 \qquad (2)$$

$$qA - pB \leq c \qquad (qA - pB - c)x = 0 \qquad (3)$$

$$p,\ q,\ x \geq 0$$

Their interpretation is the following: (3) says that profit must be nonpositive and only those activities with zero profit are operated at positive level; (1) says that demand must be satisfied and the price of a product is positive only if there is no excess of that product; (2) says that resources are in limited quantities and the rent accrued to a resource is positive only if there is no excess supply.

The dual of (SSLP) involves maximization of the revenue subject to the non-profit condition.

### 3.2.2 Non-Linear Equilibrium Models

Models of this type have been less extensively used than minimum-cost LPs, probably due to the greater difficulty in solving non-linear programs than linear programs and to the difficulty of estimating demand and supply functions (lack of data), but their importance is more and more recognized.* The advantages of incorporating demand functions in planning models are multiple: first, it simulates a competitive market equilibrium; second, it incorporates more substitution possibilities in the model; third, it allows analysis of the impact of pricing policies, such as subsidies and taxes on products and factor prices.

The notations are the same as before:

$$\text{(SSEP)} \quad \begin{aligned} &\text{Max } f(d) - g(s) - cx \\ &Ax - d \geq 0 \\ &Bx - s \leq 0 \\ &x, d, s \geq 0 \end{aligned}$$

Here again the elasticity of supply of some commodities could be zero:

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* For a statement of this fact in energy modeling, see Hoffman and Wood [19]. The world oil model of Kennedy [22] is an example of a static equilibrium model in energy planning.

$$\text{Max } f(d) - g(s) - cx$$
$$Ax - d \geq 0$$
$$B^1 x - s \leq 0$$
$$B^2 x \leq \overline{s}$$
$$x, d, s \geq 0$$

For instance, in an agriculture model, land and water could be available in limited quantity defined exogenously and the elasticity of supply for them would be zero. But labor supply might be given by a supply curve with elasticity different from zero and from infinity.

The Kuhn-Tucker conditions for (SSEP) are:

$$Ax - d \geq 0 \qquad (Ax - d)q = 0 \tag{4}$$
$$Bx - s \leq 0 \qquad (Bx - s)p = 0 \tag{5}$$
$$-qA + pB \geq -c \qquad (qA - pB - c)x = 0 \tag{6}$$
$$q \geq \nabla f(d) = \phi(d) \qquad (q - \phi(d))d = 0 \tag{7}$$
$$p \leq \nabla g(s) = \psi(s) \qquad (p - \psi(s))s = 0 \tag{8}$$
$$x, d, s, p, q \geq 0$$

(4), (5) and (6) are the same as (1), (2) and (3); (7) is the consumer equilibrium condition: the shadow prices for the demand constraints must be greater than or equal to the demand prices (as defined by the demand function $\phi = \nabla f$) and demand for a product is positive only if these prices are equal; (8) is the producer equilibrium condition and its interpretation is similar to that of (7).*

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* For more on these optimality conditions and their interpretation as competitive equilibrium, see Takayama and Judge [33].

Model builders have found it convenient to combine econometric models for demand (to estimate the demand functions) with an engineering model - or activity analysis model - of supply.* This reflects the behavioral aspect of demand and the engineering aspect of supply.

### 3.3 Dynamic Models

Again, the main difference with static models is that investment activities are treated explicitly and successive periods are linked through capital accumulation. We obtain a different formulation, depending on whether the capital formation activity is endogenous to the model or exogenous. We assume the latter; in that case, the marginal cost of capital is known and given. Normally it would be obtained from an economy-wide model. Sectoral models are usually expressed in physical units rather than monetary units. $i_t$ will now be the vector of addition to capacity in period t in physical units rather than the vector of investment in monetary units. Let $r_t$ be the vector of capacity expansions costs by sector in period t (each component is the cost of adding a unit of capacity in the sector), K a diagonal matrix of capacity requirements for each activity and $\bar{k}_1$ initial capacity. The other rotations are the same except that vectors have been subscripted by t. T is the planning horizon and t = 1---T. We assume that it is given and finite and leave aside the problem of setting terminal conditions.

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* For a description of these models for energy modeling see [19]. See also [33] and [20].

$$\text{(SDLP)} \quad \text{Min} \sum_{t=1}^{T} \omega^{t-1}(c_t x_t + r_t i_t)$$

s.t.

$$Ax_t \geq d_t \qquad \text{for all t.}$$

$$Bx_t \leq s_t \qquad \text{for all t.}$$

$$Kx_t \leq \bar{k}_1 + \sum_{\tau=1}^{t-1} i_\tau \qquad \text{for all t.}$$

$$x,\ i \geq 0$$

The remarks made for the supply side of the static LP are applicable here also. Another dynamic component could be introduced with storage activities and stockpile variables, for final commodities and primary commodities. In that case, a term could be added to the objective function corresponding to the cost of carrying inventories from one period to the next.* This extension could also be made on the dynamic equilibrium model which follows:

$$\text{(SDEP)} \quad \text{Max} \sum_{t=1}^{T} \omega^{t-1}(f_t(d_t) - g_t(s_t) - c_t x_t - r_t i_t)$$

$$Ax_t - d_t \geq 0 \qquad \text{For all t.}$$

$$Bx_t - s_t \leq 0 \qquad \text{For all t.}$$

$$Kx_t - \sum_{\tau=1}^{t-1} i_\tau \leq \bar{k}_1 \qquad \text{For all t.}$$

$$x,\ i,\ d,\ s \geq 0$$

For the dynamic models, we notice that matrices A, B and K could be subscripted with the time index if they are assumed

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* For a description of this and interpretation of optimality conditions, see Takayama and Judge [33].

to vary over time (but they must be known for the all planning horizon).

*Let the dual variables be respectively* $\omega^{t-1}q_t$, $\omega^{t-1}p_t$, $\omega^{t-1}v_t$. The optimality conditions for (SDLP) are as follows:

$$Ax_t \geq d_t \qquad (Ax_t - d_t)q_t = 0 \qquad \text{For all } t. \tag{1}$$

$$Bx_t \leq s_t \qquad (Bx_t - s_t)p_t = 0 \qquad \text{For all } t. \tag{2}$$

$$Kx_t - \sum_{\tau=1}^{t-1} i_\tau \leq \bar{k}_1 \qquad (Kx_t - \sum_{\tau=1}^{t-1} i_\tau - \bar{k}_1)r_t = 0 \quad \text{For all } t. \tag{3}$$

$$q_tA - p_tB - v_tK \leq c_t \quad (q_tA - p_tB - v_tK - c_t)x_t = 0 \quad \text{For all } t. \tag{4}$$

$$\sum_{\tau=t+1}^{T} \omega^{\tau-t} v_\tau \leq r_t \qquad (\sum_{\tau=t+1}^{T} \omega^{\tau-t} v_\tau - r_t)i_t = 0 \qquad \text{For all } t. \tag{5}$$

(1), (2) and (4) for each time period are similar to the corresponding conditions for the static model and their interpretation is the same; (5) says that the investment cost in period t must be greater than or equal to the discounted sum of the rents accrued to capacity for year t+1 on, and investment is positive in year t, only if they are equal; (3) says that production activities cannot use more capacity than is available in year t and the rental cost accrued to capacity is positive only if capacity is fully utilized.

Similarly, the Kuhn-Tucker conditions for (SDEP) are:

(1), (2), (3), (4), (5) and

$$q_t \geq \nabla f_t(d_t) = \phi_t(d_t) \qquad (q_t - \nabla f_t(d_t))d_t = 0 \quad \text{For all } t. \tag{6}$$

$$p_t \leq \nabla g_t(s_t) = \psi_t(s_t) \qquad (p_t - \nabla g_t(s_t))s_t = 0 \quad \text{For all } t. \tag{7}$$

By now, their interpretation should be clear.

As always with an intertemporal model, there arises the problem of how to choose a planning horizon and how to set terminal conditions. We could repeat here the type of discussion conducted for dynamic economy-wide models. The problem might even be harder in sectoral planning, as in this case there is no macroeconomic growth theory to guide these choices. The same type of devices can be used such as fixing terminal capital stock or valuing it in the objective function. But these values might be hard to estimate. A less rigid possibility would be to set terminal conditions on aggregate variables and, eventually, derive them from a multi-sector economy-wide model.

Nordhaus [30] recognizes the difficulty of choosing a planning horizon T and suggests that, if there exists a "backstop technology," i.e., a set of processes capable of meeting demand with a virtually infinite resource base, then T is the time at which transition occurs; if resources are limited and no backstop technology exists, the horizon T is the time of exhaustion. Kalymon [21] follows the same approach and, in his model, the final year T is not fixed but determined within the optimizing process. But, for a complex model with several different resources and processes, the calculation of the transition time or of the exhaustion time might be cumbersome.

### 3.4 Objective Function

In the previous paragraphs, two different objective functions were used: a cost function used as minimand and a "net payoff" function (sum of producers' and consumers' surpluses) used as maximand to describe a competitive equilibrium. A monopolist equilibrium can be represented by a model such as (SSEP) but with a different objective function; the objective function is now

$$\text{total profits} = \text{total revenue} - \text{total cost}$$
$$= \phi(d)d - \psi(s)s;$$

marginal revenue is equal to $\nabla\phi(d)d + \phi(d)$ and marginal cost to $\nabla\psi(s)s + \psi(s)$. The Kuhn-Tucker conditons are the same except that the vector $\nabla f(d) = \phi(d)$ of demand prices is replaced by the vector of marginal revenues and supply prices $\psi(s)$ by marginal cost, $\nabla\psi(s)s + \psi(s)$ [33].

The price system and allocation scheme obtained with a monopolist maximand can be quite different than the one obtained in a competitive equilibrium. Depending on the situation of the sector, one objective function or the other is more appropriate.

But, as it is hard to find perfectly competitive markets, is has been suggested to use both objective functions, keeping one as a maximand and using the other one as a constraint and, eventually, solving parametrically for that constraint. For instance in CHAC [11], Duloy and Norton use a competitive maximand and introduce several constraints for farmers' incomes and profits.

As for economy-wide planning, we could also think of alternative criteria for sectoral planning, such as pollution control, environmental quality and reduction of unemployment, to name just a few.

3.5 Spatial Models

One of the possible dimensions of sectoral models is the spatial dimension. In some cases, it is important to use a geographically disaggregated model, so as, for instance, to use efficiently the productive capabilities of each region and decide on interregional allocation. The efficient location of new facilities can be analyzed only with a spatial model; but, for this purpose, a very detailed model is needed, and this problem is better considered part of project analysis and we shall return to it in the next section. Here we assume the locations of the plant facilities are known and fixed. In the previous models, the spatial dimension might have been included but it was not apparent. Here we make the explicit assumption that the economy is composed of L regions; $l$ and $h = 1....L$ are the indices corresponding to regions. Transportation activities play a very important role here. There are several possible treatments of the transportation sector: it can be exogenous to the model; in that case, it appears in the objective function and right hand side of the model in the form of cost coefficients and exogenously set capacity limits. It can be endogenous to the model (which in that case encompasses several sectors), and be specified, for instance, by its own input-output matrix; increases to capacity in the transportation

sector can then be decided endogenously to the model. Or a separate model can be designed for the transportation sector, allowing for more complex specification of the transportation network and cost structures. We adopt here the first approach. Final commodities are all assumed to be transportable between regions at a given cost. Primary commodities can either be mobile or immobile.

We add the following notations to the previous ones:

$z_{lh}$ n vector of quantities of final commodities transported from region $l$ to region h.

$z = (z_{lh})$ all $l$, all h

$y_{lh}$ quantity of mobile primary commodities transported from $l$ to h.

$y = (y_{lh})$ all $l$, all h

$t^f_{lh}$ unit transport cost for shipment of final commodities from $l$ to h.

$t^f = (t^f_{lh})$

$t^{mp}_{lh}$ unit transport cost for shipment of mobile primary commodities from $l$ to h.

$t^{mp} = (t^{mp}_{lh})$

The other vectors and matrices are now subscripted by an index $l$; for instance, $d_l$ is demand in region $l$ and $A_l$ is the output matrix for region $l$. The input matrix $B_l$ is partitioned $B_l = \begin{pmatrix} B^{im}_l \\ B^m_l \end{pmatrix}$. $B^{im}_l$ is the input matrix for immobile primary resources for region $l$ and $B^m_l$ for mobile factors.

We first formulate a spatial minimum cost linear model of production and allocation:

$$\text{(SSSLP)} \quad \text{Min} \sum_{l=1}^{L} c_l x_l + t^f z + t^{mp} y$$

$$A_l x_l - \sum_{\substack{h=1\\h\neq l}}^{L} z_{lh} + \sum_{\substack{h=1\\h\neq l}}^{L} z_{hl} \geq d \qquad l = 1\text{---}L$$

$$B_l^{im} s_l \leq s_l^{im} \qquad l = 1\text{---}L$$

$$B_l^{m} x_l + \sum_{\substack{h=1\\h\neq l}}^{L} y_{lh} - \sum_{\substack{h=1\\h\neq l}}^{L} y_{hl} \leq s_l^{m} \qquad l = 1\text{---}L$$

$$x,\ y,\ z \geq 0$$

As was mentioned before, additional constraints could be introduced such as upper bounds on some of the quantities $y_{lh}$ and $z_{hl}$, reflecting capacity limitations on the transportation network.

Let the vectors of dual variables respectively for each region $l$ be $q_l$, $p_l^{im}$ and $p_l^{m}$. The optimality conditions for (SSSLP) are:

$$A_l x_l - \sum_{\substack{h=1\\h\neq l}}^{L} z_{lh} + \sum_{\substack{h=1\\h\neq l}}^{L} z_{hl} \geq d_l \qquad (A_l x_l - \sum_{\substack{h=1\\h\neq l}}^{L} z_{lh} + \sum_{\substack{h=1\\h\neq l}}^{L} y_{hl} - d_l) q_l = 0$$

For all $l$. (1)

$$B_l^{im} x_l \leq s_l^{im} \qquad (B_l^{im} x_l - s_l^{im}) p_l^{im} = 0$$

For all $l$. (2)

$$B_l^{m} x_l + \sum_{\substack{h=1\\h\neq l}}^{L} y_{lh} - \sum_{\substack{h=1\\h\neq l}}^{L} y_{hl} \leq s_l^{m} \qquad (B_l^{m} x_l + \sum_{\substack{h=1\\h\neq l}}^{L} y_{lh} - \sum_{\substack{h=1\\h\neq l}}^{L} y_{hl} - s_l^{m}) p_l^{m} = 0$$

For all $l$. (3)

$$q_l A_l - p_l B_l \leq c_l \qquad (q_l A_l - p_l B_l - c_l) x_l = 0$$

For all $l$. (4)

$$-q_l + q_h \leq t^f_{lh} \qquad (-q_l + q_h - t^f_{lh})z_{lh} = 0$$

For all $l$ and h. (5)

$$-p^m_l + p^m_h \leq t^{mp}_{lh} \qquad (-p^m_l + p^m_h - t^{mp}_{lh})y_{lh} = 0$$

For all $l$ and h. (6)

$$x,\ y,\ z,\ q,\ p \geq 0$$

The interpretation of (1), (2), (3) and (4) is straightforward. (5) and (6) are the locational equilibrium conditions. (5) says that the demand price of final commodities in region h is less than or equal to the price of the commodities in region $l$ plus the transportation cost between $l$ and h and only commodities for which equality holds are transported from $l$ to h. (6) is similar for mobile primary commodities.

Next we formulate a spatial equilibrium model. It is assumed that demand and supply functions $\phi_l$ and $\psi_l$ are known for each region, that a welfare function can be defined for each region, and that the community welfare function is additive and equal to the sum of the welfare of each region.

$$\text{Max} \sum_{l=1}^{L}(f_l(d_l) - g_l(s_l) - c_l x_l) - t^f z - t^{mp} y$$

$$A_l x_l - \sum_{\substack{h=1 \\ h \neq l}}^{L} z_{lh} + \sum_{\substack{h=1 \\ h \neq l}}^{L} z_{hl} - d_l \geq 0 \qquad l = 1\text{---}L$$

$$B^{im}_l x_l - s^{im}_l \leq 0 \qquad l = 1\text{---}L$$

$$B^m_l x_l + \sum_{\substack{h=1 \\ h \neq l}}^{L} y_{lh} - \sum_{\substack{h=1 \\ h \neq l}}^{L} y_{hl} - s^m_l \leq 0 \qquad l = 1\text{---}L$$

$$x,\ y,\ z,\ s,\ d \geq 0$$

These models could also contain the time dimension. In that case, the model can become quite complex and its size gets large. Intertemporal and spatial models can be very useful in planning the location and timing of investments. To keep the problem manageable, the scope of the model is narrowed. dealing only with part of the sector. An example of this is INTERCON [6] which will be discussed in the next section on project selection.

# 4. MODELS FOR PROJECT SELECTION AND EVALUATION

## 4.1 Introduction

The role of project analysis is to decide, from among a set of predesigned projects, which ones are profitable and which ones should be implemented.

Traditionally, the method used for project evaluation is cost-benefit analysis on a single project basis. The main drawback to this approach is that it does not take into account interdependencies between different projects and between a project and the rest of the economy. It assumes that the parameters used for the evaluation of the project would not change if the project was implemented. It does not allow analysis of the location and timing of the investment.* Hence, the method is suitable only if the project is small enough so that its impact on the rest of the economy can be neglected. If interdependencies and economies of scale are important, then the project evaluation should be embedded in a model of the sector (or sectors) on which the project would have a significant impact. In some cases, it might even be necessary to analyze the project through a model of the entire economy, if it would affect the value of national parameters, such as prices of foreign exchange and capital. Hence, "project choices should be embedded in a model

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* In fact, these last statements are not quite true as the project analyst tries to correct for interdependencies, but this is done in an informal way, as opposed to the more formal method of using the models described later.

whose scope is sufficiently broad that significant interdependencies do not cross its boundaries."* Before describing, in the following paragraph, each of the methods in more detail, we discuss what is meant by a project. A project is defined as the construction of a given plant in a given sector at a given location in a given time period. Building similar plants in different time periods is considered as different projects. For instance, the project analyst might be presented with several projects producing the same output, but differing by type of producing process, location and date, and he has to pick out the most profitable for the economy.

### 4.2 Cost-Benefit Analysis

This is the most traditional and extensively used method for project appraisal. As a number of good manuals and surveys** exist, it will not be reviewed in detail. Besides, in this survey, we are mainly concerned with optimization models.

Briefly, the analyst appraises the project on the basis of its internal rate of return or its net present value. He assesses the direct benefits and costs, as well as the indirect benefits and indirect costs, of each project, chooses a rate of discount, and then rank orders the projects according to their net present value. Then the most profitable projects

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* Westphal, Chapter XV in [2].

** See, for instance, UNIDO [34], Little and Mirrlees [24], Dasgupta [8] and Harberger [18].

are implemented in their order on the list until the budget is exhausted (only those with a non negative net present value are considered for implementation). If the measure used is commercial profitability of the project, then its inputs and outputs are valued at market prices. But usually this is not a proper measure for evaluation of projects by governments. National profitability seems to be a more adequate criterion. In that case, market prices have to be corrected to reflect their real worth to the entire economy. Shadow prices, or accounting prices, are then used to value inputs and outputs. They can be estimated by various indirect methods [34], but ideally, they should be provided by higher-level models. National parameters, such as the prices of foreign exchange, capital and labor, could be obtained as the dual solution of an economy-wide model. Prices of input commodities could be obtained from a sectoral model. The problem is that usually the dual variables of these models are not very reliable.* A lot of research effort, lately, has been put into developing economy-wide models that would provide more reasonable shadow prices.**

From this brief description, it appears clearly that the method is not adequate if interdependencies are important. It can be applied only if the impact of the project on the price structure is assumed to be negligible. In that case, the project analyst need only know the

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* For a discussion of this, see Bruno, Chapter VIII in [2].

** See, for example, Goreux [17].

prices at which to value inputs and outputs. Otherwise, if the project is large, it should be incorporated in a higher-level model. Besides, interrelated projects should be evaluated together to ensure proper coordination of the different investment plans.

### 4.3 Use of a Model of a Sector (or Subsector)

This approach is desirable for evaluating a project which, if implemented, would affect the prices of its inputs and outputs, and, also, for planning timing and scale of a set of interdependent projects within a sector. For example, ENERGETICOS, model of the energy sectors in Mexico [13], was used to plan simultaneously for capacity expansion in the steel, petroleum and electricity industries (the energy sectors) as "Within Mexico's energy sectors, the efficiency price of petroleum fuel is not an exogenous datum set by the international market. The price of fuel then is a source of interdependencies between project decisions."* Similarly, for Mexico, it was found desirable to evaluate simultaneously transmission projects and generating plants in the electric power sector, and thus to design for that purpose a geographically disaggregated model of this subsector, INTERCON [13].

In the sectoral models formulated in the previous section, individual capacity expansion activities reflected the high

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* Manne in [28].

degree of aggregation over investment variables. Now capacity expansion activities should represent well defined projects and reflect the indivisibility of processing facilities, i.e., capacity does not increase continuously but each plant built is an indivisible addition to capacity. Each potential project appears in the sectoral model as a vector of data - size of the plant, input requirements, output possibilities, cost of capital, etc., - and an integer variable (which will be equal to 1 in the optimal solution if the project is profitable and should be implemented). As the number of possible projects can be quite large, they are first screened out by solving a linear sectoral model without integer variables, and by analysis by experts, and the most promising ones are selected. Then a mixed integer program of the type given below is formulated where only the selected projects appear. In such models, economies of scale in capital costs can also be introduced by using integer variables.

We give here a modified formulation of (SDLP) to allow for project selection. Let us assume, for simplicity, that capacity expansion occurs in this indivisible way for each activity appearing in the model (this is not at all necessary). Let $\delta_t$ be a vector of 0 and 1; a component of $\delta_t$ is 1 if a plant is built in that sector in period t. Let M be a diagonal matrix of plant scales. If there exist economies of scale in capital cost, a fixed cost appears in the objective function. Let $f_t$ be the vector of these costs.

With these notations, the formulation is as follows:

$$\text{(PSMIP)} \quad \text{Min} \sum_{t=1}^{T} \omega^{t-1}(c_t x_t + r_t i_t + f_t \delta_t)$$

$$Ax_t \geq d_t \qquad t = 1 \ldots T$$

$$Bx_t \leq s_t \qquad t = 1 \ldots T$$

$$Kx_t \leq \bar{k}_1 + \sum_{\tau=1}^{t-1} i_\tau \qquad t = 1 \ldots T$$

$$i_t = M\delta_t$$

$x_t$, $i_t \geq 0$ for all t. $\delta_t$ binary vector.

There could be different plant scales possible in some sectors. In that case, a different integer variable should appear for each possible scale as they are considered as different projects, e.g., if in sector n there could be a choice between a plan of size $M_n$ and one of size $M'_n$, then for this sector

$$i_n = M_n \delta_n + M'_n \delta'_n.$$

If it is specified that at most one plant of each type can be built during the planning horizon, an additional constraint appears in (PSMIP):

$$\sum_{t=1}^{T} \delta_t \leq e \quad \text{(e is a vector of one (1))}$$

Other exclusivity conditions can be specified in this way for some projects. Budget constraints could also be introduced.

Though, to my knowledge, no example appears in the literature, I imagine that a sectoral equilibrium model for project evaluation and selection could be formulated in the same way.

When the spatial dimension plays an important role (as in INTERCON [13]), a geographically disaggregated sectoral model with integer variables can be formulated along the same lines.

### 4.4 Use of an Economy-Wide Model

The use of such a model might be required to reflect the impact of a project's implementation throughout the entire economy. "Westphal found, for example, that each of two large scale import substituting projects proposed in Korea, a steel mill and petrochemicals complex, required such a large faction of available investment resources and had such an enormous impact on the foreign exchange balance that the timing of the implementation of either project would have an identifiable impact on the shadow prices of investment and foreign exchange. Plans for project implementation thus had to be evaluated in an economy-wide model that specified increasing returns in the technology of both projects."* This was not the case in Mexico where, after experiments, it was found unncessary to evaluate energy projects in an economy-wide model.** Such an approach is necessary, also, when, in countries at an early stage of industrialization, a new type of industrial project represents a new sector (if implemented, it would substitute domestic production for imports). This was the case for an iron ore mine in the Ivory Coast***

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* Westphal in [2]. See also [35].

** Manne in [28].

*** Goreux [17].

(because if the project was not implemented, the country would not be producing any iron ore), and for the petrochemicals complex in Korea.* These projects certainly have to be evaluated with a multi-sector economy-wide model. But decisions at such low levels as project selection require a lot of detailed information. Westphal found it convenient to introduce details on the two projects and the sectors directly related to them and to aggregate the rest of the economy in just a few sectors. The formulation he adopted is very similar to theone given below. The notations are the same as in (MDP) with the following addition: $\delta_t$ is a vector of binary variables with $\delta_{it} = 1$ if a plant is built in sector i in period t; $\overline{B}$ is an n x n matrix of fixed requirements capacity input coefficients (for simplicity, we assume that there are fixed investment requirements for capacity expansion in each sector. In Westphal's model, this is true only for two sectors.) V is an n x n diagonal matrix of plant scales.

$$\text{(PMMTP)} \quad \text{Max} \sum_{t-1}^{T} \omega^{t-1} c_t$$

$$(I - A)x_t \geq \xi_t c_t + \overline{B}\delta_t + Bi_t \qquad t = 1\text{---}T$$

$$Kx_t \leq \overline{k}_1 + \sum_{\tau=1}^{t-1} i_\tau \qquad t = 1\text{---}T$$

$$i_t \leq V\delta_t$$

$$x_t,\ i_t,\ c_t \geq 0 \text{ for all } t. \quad \delta_t \text{ is a binary vector.}$$

Of course, terminal conditions should be specified, and all the types of extensions mentioned for economy-wide models could be considered here.

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* Westphal [35].

4.5 Conclusions

In conclusion, we note that there exist other tools for project appraisal than the traditional cost-benefit analysis. Which one is most appropriate depends on the project(s) to be evaluated and on the economy in which it is implemented. Sectoral or economy-wide models are more complex to formulate, more difficult to solve (especially MIPs) and may require more data than cost-benefit analysis. Hence, it is preferable to use cost-benefit analysis when the results obtained are satisfactory, i.e., when interdepencies can be neglected. But this is something hard to decide *a priori* - it can be done only after solving the different models and comparing the results, or conducting similar experiments (see, for instance, [29]). But Westphal noted in [2] that several rules of thumb emerge: price interdependencies are more likely in a highly protected economy than in a competitive economy that engages in free international trade, in developing economies than in economies which have reached a high degree of industrialization. All other things being equal, economies of scale and interdependencies are probably more significant in small countries than in large countries. The scale of the project is also an important factor. "The larger a project is, either in relation to the markets for its inputs and outputs or in relation to its use of available investment resources, the more far reaching will be the effects of its implementation. ... The absolute size of the economy may be the most important determinant of a given project's impact on relative shadow prices. For example, there is a big difference between building the first steel

mill in Korea, the fourth or fifth in India or the fiftieth in the United States. Adequate planning of the steel mill may require an economy-wide model in Korea, a sector model in India, and no more than project appraisal in the United States. To summarize: what must be planned at the economy-wide or regional level in some countries may be equally well planned at the sector or enterprise level in other countries." A lot more research has to be conducted for more rigorous rules to emerge. At the moment, a great deal of judgment and expertise are necessary to select the suitable approach.

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APPENDIX I

In this Appendix, we illustrate the so-called "flip-flop" behavior of the solution to linear planning models. We study here a simplified version of (MDP) defined in section 9.2:

$$\text{Max} \sum_{t=1}^{T} \omega^{t-1} c_t$$

$$\text{s.t. } (I - A)x_t \geq \xi_t c_t + Bi_t \tag{1}$$

$$Kx_t \leq \bar{k}_1 + \sum_{\tau=1}^{t-1} i_\tau \tag{2}$$

$$x_t,\ c_t,\ i_t \geq 0$$

The primary resources constraints have been dropped from the formulation of (MDP). The feasibility set is non empty and, at optimum, constraints (1) are satisfied with equality.

The solution to this problem displays a "flip-flop" behavior in the sense that investment is concentrated in the first periods and consumption in the last ones. The period in which the change occurs depends on the value of the discount rate $\omega$.

We first illustrate this on a three periods, one sector model:

$$\text{Max } c_1 + \omega c_2 + \omega^2 c_3$$

$$\text{s.t. } (1 - a)x_1 \geq c_1 + bi_1$$

$$(1 - a)x_2 \geq c_2 + bi_2$$

$$(1 - a)x_3 \geq c_3 + bi_3$$

$$Kx_1 \leq \bar{k}_1$$

$$Kx_2 \leq \bar{k}_1 + i_1$$

$$Kx_3 \leq \bar{k}_1 + i_1 + i_2$$

All the constraints are satisfied with equality at optimum and we reformulate the problem as:

$$
\begin{aligned}
\text{Max}\quad & \omega c_1 + \omega c_2 + \omega^2 c_3 \\
\text{s.t.}\quad & c_1 + b i_1 = k \\
& c_2 - \alpha i_1 + b i_2 = k \\
& c_3 - \alpha i_1 - \alpha i_2 = k \\
& \text{non-negativity}
\end{aligned}
$$

where

$$k = (1 - a)\frac{\bar{K}_1}{K} \quad \text{and} \quad \alpha = \frac{1 - a}{k}$$

Consider the basis $\{c_1, c_2, c_3\}$. The corresponding matrix is $B = I$. The dual variables are:

$$\pi = (1, \omega, \omega^2)$$

Then the reduced cost vector is:

$$\bar{c} = (0, 0, 0, -b + \alpha\omega(1 + \omega), - b\omega + \alpha\omega^2)$$

This basis is optimal for

$$\omega \leq \frac{b}{\alpha} \qquad \text{and} \qquad \omega(1 + \omega) \leq \frac{b}{\alpha}$$

or $\quad \omega \leq \omega(1 + \omega) \leq \dfrac{b}{\alpha}$

i.e., for small values of $\omega$.

In this case, the optimal solution is:

$$c_1 = c_2 = c_3 = k$$

$$i_1 = i_2 = 0$$

Investment is equal to zero and consumption remains constant during the planning period. When $\omega$ increases and becomes such that

$\omega(1 + \omega) = \frac{b}{\alpha}$, $i_1$ enters the basis and $c_1$ leaves. The new basis is:

$$B = \begin{pmatrix} 0 & 0 & b \\ 1 & 0 & -\alpha \\ 0 & 1 & -\alpha \end{pmatrix}$$

hence $\pi = (\omega(1 + \omega)\frac{\alpha}{b}, \omega, \omega^2)$

and $\bar{c} = (1 - \omega(1 + \omega)\frac{\alpha}{b}, 0, 0, 0, -b\omega + \alpha\omega^2)$

This basis is optimal for

$$\omega \leq \frac{b}{\alpha} \leq \omega(1 + \omega)$$

The optimal solution is:

$$c_1 = 0 \qquad\qquad i_1 = \frac{k}{b}$$

$$c_2 = c_3 = k + \frac{\alpha}{b}k \qquad\qquad i_2 = 0$$

When $\omega$ increases and becomes equal to $\frac{b}{\alpha}$, then $i_2$ enters the basis and $c_2$ leaves. The optimal solution is:

$$c_1 = 0 \qquad\qquad i_1 = \frac{k}{b} \qquad\qquad i_2 = \frac{k}{b} + \alpha\frac{k}{b^2}$$

$$c_3 = k(1 + \frac{\alpha}{b})^2$$

In this last case, consumption is all concentrated in the last period.

We now examine the T periods one sector problem:

$$\text{Max} \quad c_1 + \omega c_2 + \dots + \omega^{T-1} c_T$$

$$c_1 + bi_1 = k$$

$$c_2 - \alpha i_1 - \alpha i_2 + bi_3 = k$$

- - - - - - - - - - - - - - -

$$c_t - \alpha i_1 - \alpha i_2 \dots - \alpha i_{T-1} = k$$

non negativity

We claim that there exists $T_o$, $1 \leq T_o \leq T$, such that, in an optimal solution:

$$c_t = 0 \qquad i_t \neq 0 \qquad \text{for } t < T_o$$

$$c_t \neq 0 \qquad i_t = 0 \qquad \text{for } t \geq T_o$$

Proof:

Suppose there exists $t_o$ such that

$$C_{t_o} \neq 0 \qquad i_{t_o} = 0$$

$$c_{t_{o+1}} = 0 \qquad i_{t_{o+1}} \neq 0$$

let $c'_{to} = 0$ $\qquad i'_{to} = \frac{c_{to}}{b}$

$$c'_{t_o+1} = c_{t_o} + i'_{t_o} \qquad i'_{t_o} = i_{t_o+1}$$

$$= c_{t_o} + \frac{\alpha}{b}c_{t_o}$$

$$\left.\begin{array}{l} c'_t = c_t \\ i'_t = i_t \end{array}\right. \quad \text{for } t \neq t_o,\ t_o+1$$

Thus the difference between the values of the objective function for the supposedly optimal solution and the new feasible solution is:

$$\omega^{t_o-1} c_{t_o} - \omega^{t_o}(t_o + \frac{\alpha}{b}c_{t_o}) = \omega^{t_o-1} c_{t_o}[1 - \omega(1 + \frac{\alpha}{b})]$$

This is negative for values of $\omega$ greater than $\frac{b}{b+\alpha}$. Hence, in this case, it is better to invest earlier and to consume later. Consumption is delayed in this way until it is concentrated in the last periods of the plan.

In fact, the switching time $T_o$ is defined by

$$(1 + \omega + \ldots + \omega^{T-T_o}) > \frac{b}{\alpha} \geq (1 + \omega + \ldots + \omega^{T-T_o-1})$$

and investment occurs in the first To -1 periods and consumption in the last periods.

If $\omega = 1$ (consumption in each period is equally valued),

$$T - T_{o+1} > \frac{b}{\alpha} \geq T - T_o \qquad \text{or} \qquad T - T_o = [\frac{b}{\alpha}]$$

and no investment occurs in the last $[\frac{b}{\alpha}]$ periods.

It is this type of behavior, illustrated here for the one sector model, that is qualified as "flip-flop" or "bang-bang." The multisectoral model presents the same characteristics, but, in this case, it is more complex to illustrate.

This behavior is due to several simplifications introduced in the present model. First, *no terminal conditions have been* specified in the simplified model, thus not taking into account the post-plan period.

Introducing terminal conditions, as suggested in section 2.3, would force investment to occur in the last periods of the plan. But this would not fundamentally change the "flip-flop" behavior of the model, which is basically due to the linearity of the objective function. Introducing primary resource constraints will slightly correct this behavior, without really suppressing it. Where these constraints are not binding, the model produces the same type of "bang-bang" solution. To avoid this, with a linear welfare function, some kind of smoothing devices, as described in section 2.2, should be used.

## APPENDIX II

### Defining a Sectoral Objective Function: Consumer's Surplus, Integrability Conditions and Convexity

The intent of this short note is just to clarify the mathematical notion of consumer's surplus and the somewhat confused issue of the necessity of the integrability conditions. Most of the economic rationale is not given here and can be found in the references.

### Consumer's Surplus

The concept of consumer's surplus has been used widely to formulate economic equilibrium models and price determination models. It represents the difference between what a consumer would be willing to pay for a good and what he actually pays, thus it is a measure of the net benefit to the consumer for buying some quantity of a good. Gross benefit is the sum of consumer's surplus and total revenue. Hence, to represent the goal of a firm or sector in a model where demands and prices are both endogenous, a suitable and easily quantifiable objective function to maximize is:

total revenue + consumer's surplus - total cost

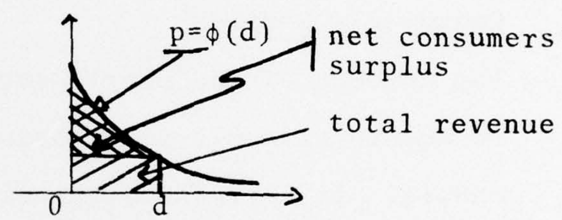
But in the multigood case with interdependent demands, the consumer's surplus cannot be defined for arbitrarily given demand functions. It will exist if the consumer behavior is assumed utility maximizing. In that case the consumer's surplus is proportional to the utility function of the consumer. The utility maximization hypothesis imposes certain observable

restrictions on consumer behavior. These will be discussed in more detail after a mathematical treatment of consumers' surplus.

Mathematical Formulation*

For a single good, the demand curve is given as in Figure 1 and the gross consumer's surplus S (net consumer's surplus + total revenue) is equal to the area under the demand curve:

$$S(d) = \int_0^d \phi(\delta)\,d\delta$$




In the multi-good case where the demand functions are independent, a similar quantity can be defined for each individual good and the total consumer's surplus is just the sum of the individual surpluses. We would like to define a similar quantity in the multi-good case with interdependent demands. The problem is the following: given a function $\phi$ from $R^n$ into $R^n$ which associates a price vector to a demand vector, find a function S from $R^n$ into R such that

$$\nabla S = \phi$$

(The Jacobian of S is equal to $\phi$.)

In Physics or Mathematical Analysis [1], the function S is called the potential function of the vector field $\phi$. If $\phi$ is continuously differentiable, and has a potential function,

* The mathematical theorems can be found in Apostol [1].

then we must have:

$$\frac{\partial \phi_i}{\partial p_j}(p) = \frac{\partial \phi_j}{\partial p_i}(p) \text{ for all p. } i \neq j,\ i, j = 1....n$$

These conditions are known in the economic literature as the integrability conditions.* They are necessary and sufficient for the line integral $\int_{\Gamma(o,d)} \phi(\delta)d\delta$ to exist, to be well defined and independent of the path of integration. In that case the quantity S is well defined and we can write

$$S = \int_o^d \phi(\delta)d\delta$$

where the path of integration is not specified. This is defined as

$$S = \int_o^{d_1} \phi_1(\delta_1, d_2, \ldots, d_n)d\delta_1$$

$$+ \int_o^{d_2} \phi_2 (0, \delta_2, d_3, ---, d_n)d\delta_2$$

$$+ \ldots.$$

$$+ \int_o^{d_n} \phi_n(0, 0, \ldots, 0, \delta_n)d\delta_n$$

It is immediate to verify that S thus defined satisfy:

$$\nabla S = \phi$$

Integrability Conditions

If the integrability conditions are not satisfied by $\phi$, no potential function exists for $\phi$, hence the consumer's surplus

---

* Often, demand functions are assumed linear, i.e., $\phi(d) = Qd + r$. In that case the integrability conditions just say that Q must be symmetric. Besides S will be concave if and only if Q is negative semidefinite.

cannot be defined, as the previous integral always depends on the path of integration. The utility maximization hypothesis is strictly equivalent to the integrability conditons,* i.e., the matrix of substitution effects must be symmetric negative semidefinite (see Hurwicz and Uzawa [2]).

The integrability conditions have been discussed greatly in the economic literature since 1930. They have been proved to be equivalent to the strong axioms of revealed preferences [9], [5] and have been justified by several economic arguments. These will not be discussed here but can be found in the references.

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* In the mathematical formulation above, it was assumed that $\phi$ was a function, i.e., a one to one mapping, and also that it was continuously differentiable. In fact, demand relations are in some cases point to set mappings, also called correspondences. The consumer theory has been extended to deal with demand correspondences rather than demand functions, and not to rely on assumptions on continuity and differentiability [2], [6]. The mathematical problem now is to find a closed proper concave function such that its subdifferential be $\phi$ (for definition and a rigorous treatment, see Rockafellar [8]). For this function to exist, it is necessary and sufficient that $\phi$ be maximal cyclically monotone (Th 24.9in[8]). This is as restrictive as the integrability conditions. For instance, when $\phi$ is linear from $R^n$ into $R^n$, $\phi$ is cyclically monotone if and only if Q is symetric negative semidefinite. The

In this note, we did not take into account the budget restrictions and income effects. When a fixed budget has to be met, the integrability conditions are slightly more complex as income effects appear, but similar relations must also hold (see [2], [3]).

In any case, when demand functions are estimated econometrically, without restrictions, *either they satisfy the integrability* conditions and a welfare function can be defined or they do not and the economic equilibrium problem cannot be formulated as a welfare maximum problem. A possibility sometimes used is to estimate demand functions with the added restrictions that they satisfy the integrability conditions.

Concavity of the Objective Function

In section 3.2, the welfare function is defined as

$$w(d, s) = \int_0^d \phi(\delta)d\delta - \int_0^s \psi(\xi)d\xi$$

If $w$ is twice continuously differentiable (i.e., if $\phi$ and $\psi$ are continuously differentiable), then $w$ is concave if and only if its Hessian matrix

$$\begin{pmatrix} \nabla\phi(d) & 0 \\ 0 & -\nabla\psi(s) \end{pmatrix}$$

---

* (Continued from previous page) maximum cyclic monoticity is just a generalization of the integrability conditions when dealing with correspondences rather than functions.

is negative semidefinite, which is equivalent to $\nabla\phi(d)$ and $\nabla\psi(s)$ being negative semidefinite.*

In section 3.4, the monopolist objective function is defined as

$$w(d, s) = d\phi(d) - s\psi(s)$$

If $\phi$ and $\psi$ are twice continuously differentiable, then w is also twice continuously differentiable and its Hessian is

$$\begin{pmatrix} (H_\phi(d)d + 2\nabla\phi(d) & 0 \\ 0 & -H_\psi(s)s - 2\nabla\psi(s) \end{pmatrix}$$

For w to be concave, the matrix $H_\phi(d)d + 2\nabla\phi(d)$ must be negative semidefinite and $H_\psi(s)s + 2\nabla\psi(s)$ must be positive semidefinite. If the demand functions $\phi$ and the supply functions $\psi$ are linear, this is equivalent to saying that the corresponding matrices are respectively negative semidefinite and positive semidefinite.

---

* $\nabla\phi$ is the matrix of derivatives of $\phi$

$$\nabla\phi = \left(\frac{\partial\phi_i}{\partial p_j}\right)$$

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